CHAPTER 11

THE MATHEMATICS OF THE INFINITE.

THE MOST FAMILIAR EXAMPLE OF AN INFINITE COLLECTION in mathematics is the sequence of positive integers 1, 2, 3,... . There are many others, for example, the collection of all rational numbers, the collection of all circles in the plane, the collection of all spheres in space, etc. The idea of the infinite is implicit in many mathematical concepts, but it was not until the nineteenth century that the mathematical infinite became the subject of precise analysis. The first steps in this direction were taken by *Bernard Bolzano* (1781–1848) in the first half of that century, but his work went largely unnoticed at that time. The modern theory of the mathematical infinite—*set theory*—was created by Cantor in the latter half of the century. Although set theory initially encountered certain obstacles—which we shall discuss later—it has come to penetrate, and influence decisively, virtually every area of mathematics. It also plays a central role in the logical and philosophical foundations of mathematics.

The basic concept of Cantor's theory is that of *set* or *totality*, which in 1895 he defined in the following way:

> By a *set* we understand every collection to a whole of definite, well-differentiated objects of our intuition or thought.

Another possible definition of a set is that it is a collection of objects, called the *elements or members* of the set, defined by some explicit rule—typically, the possession of a prescribed property—which specifies exactly which objects belong to the collection. All the collections mentioned above are sets in this sense. There is a convenient notation for sets defined in this way. Suppose that P is a property, and write $P(x)$ for the assertion that the object x has the property P. Then

$$\{x: P(x)\}$$

denotes the set whose members are exactly those objects having the property P. In set theory it is also customary to write

$$a \in A$$ [1]

for *the object a is a member of the set A,* and

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[1] Here the symbol "∈" is a form of the Greek letter epsilon, the initial letter of the word *esti,* "is". This usage was introduced by Peano.

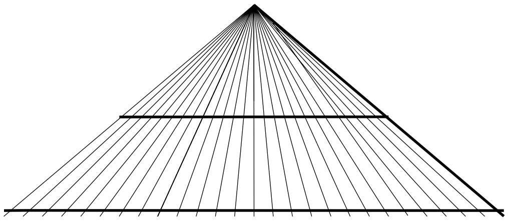
$$a \notin A$$

in the contrary case. Clearly, then, for any object *a*,

$$a \in \{x: P(x)\} \text{ if and only if } P(a).$$

More loosely, if we are given objects *a, b, c, ...,* we write {*a, b, c, ...*} for the set whose members are *a, b, c, ...*.

A central concept of set theory is that of the *equivalence* of sets, a notion which gives precise expression to the idea of two sets *having the same size*. If the elements of two sets *A* and *B* may be paired with each other in such a way that to each element of *A* there corresponds exactly one element of *B*, and vice-versa, then, as we recall, the resulting function between *A* and *B* is said to be *biunique*, and we shall say that the two sets *A* and *B* are *equivalent*. The notion of equivalence for *finite* sets clearly corresponds to the usual concept of *equality of number,* since two finite sets have the same number of elements precisely when the elements of the two sets can be put into biunique correspondence. This is just an extension of the idea of counting, because in counting the elements of a finite set we place them in biunique correspondence with a set of number symbols 1, 2, 3, ..., *n*. Cantor's idea was to extend the concept of equivalence to *infinite* sets, so as to enable them to be compared in size.



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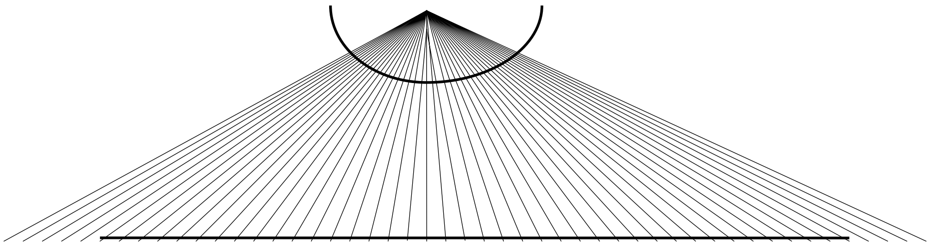

(2)

For infinite sets the idea of equivalence can lead to surprising results. For example, we observe that the sets of points on any pair of line segments are equivalent, even if the line segments are of different lengths. This can be seen by taking two arbitrary line segments and projecting from a point, as in figure (1) above. By bending one of the segments into a semicircle, as in figure (2), a similar procedure shows that this equivalence obtains even if one of the segments is replaced by the entire line. The *even numbers* form a proper subset of the set of *all natural numbers*, and these form a proper subset of the set of all (positive) *rational numbers.* (Here by a *proper* subset of a set S we mean a set consisting of some, but not all, of the objects in S.) Evidently, if a set is *finite*, i.e., if it contains n elements for some natural number n, then it cannot be equivalent to any of its proper subsets, since any such subset would contain at most $n - 1$ elements. But *an infinite set can be equivalent to a proper subset of itself.* For example, the pairing

$$\begin{array}{ccccc} 1 & 2 & 3 \ldots & & n \ldots \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ 2 & 3 & 4 \ldots & & n+1 \ldots \end{array}$$

establishes a biunique correspondence between the set of natural numbers and its proper subset of all natural numbers greater than 1, so that the two sets are equivalent. And the pairing

$$\begin{array}{ccccc} 1 & 2 & 3 \ldots & & n \ldots \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ 2 & 4 & 6 \ldots & & 2n \ldots \end{array}$$

establishes a biunique correspondence between the set of all natural numbers and the proper subset of even numbers, showing that these two sets are equivalent. It is no accident that the set of natural numbers is equivalent to some of its proper subsets, since, as we shall see, this property is *characteristic* of infinite sets.

To prove this we first observe that obviously no *finite* set can be equivalent to any of its proper subsets. Now suppose that S is an *infinite* set. We choose a member s_0 of S. Since S is infinite, we can select a member s_1 of S different from s_0, and then, for the same reason, a member s_2 of S different from both s_0 and s_1, etc. Proceeding in this way, we generate a subset $\{s_0, s_1, s_2, \ldots\} = S'$ of S. Now let $f: S \to S$ be the function defined by setting $f(s_\mathrm{n}) = s_{n+1}$ for any natural number n, and $f(x) = x$ if $x \notin S'$. It is easy to check that f is a bijection between S and its proper subset $S - \{s_0\}$. Since s_0 was arbitrary, we see that each infinite set is equivalent to any of its subsets obtained by deleting one element.

The counterintuitive nature of the equivalence between infinite sets and proper subsets is strikingly conveyed by a fable attributed to the German mathematician *David Hilbert* (1862–1943). In Hilbert's tale, he finds himself the manager of a vast hotel, so vast, in fact, that it has an *infinite* number of rooms. Thus the hotel has a first, second,..., n^{th},... room, *ad infinitum*. At the height of the tourist season, Hilbert's hotel is full: each room is occupied. (We are of course assuming the existence of an infinite number of occupants.) Now a newcomer seeking accommodation shows up. “Alas,” says Hilbert, “I have not a room to spare.” But the newcomer is desperate, and at that point an idea occurs to Hilbert. He asks the occupant of each room to move to the next; thus the occupant of room 1 is to move to room 2, that of room 2 to room 3, and so on up the line. This leaves the original occupants housed (one to a room, as before), only now the first room is vacant, and the relieved newcomer duly takes possession.

The fable does not end here, however. Suddenly a vast assembly of tourists desiring accommodation descends on the hotel. A quick tally reveals that the assembly is *infinite*, causing Hilbert some consternation. But now another idea occurs to him. This time he requests each occupant to move to the room with *double* the number of the one presently occupied: thus the occupant of room 1 is to proceed to room 2, that of room 2 to room 4, etc. The result is again to leave all the original occupants housed, only now each member of the infinite set of rooms carrying *odd numbers* has become vacant. Thus every newcomer can be accommodated: the first in room 1, the second in room 3, the third in room 5, etc. It is clear that this procedure can be repeated indefinitely, enabling an infinite number of infinite assemblies of tourists to be housed.

Hilbert's fable shows that infinite sets are intriguingly *paradoxical*, but not that they are *contradictory.* Indeed if, for example, the physical universe contains infinitely many stars—a possibility which Newton, for one, was perfectly happy to accept—then it can fill the role of "Hilbert's hotel," with the stars (or orbiting planets) as "rooms".

One of Cantor's first discoveries in his exploration of the infinite was that the set of *rational numbers*—which includes the infinite set of natural numbers and is therefore itself infinite—is actually *equivalent* to the set of natural numbers. At first sight it seems strange that the densely arranged set of rational numbers should be similar to its discretely arranged subset of natural numbers. For, after all, the (positive) rational numbers cannot be discretely arranged in *order of magnitude*: starting with 0 as the first positive rational, we cannot even choose a second "next larger" rational because, for every such choice, there will always be one smaller. Cantor observed that, nevertheless, if we *disregard* the relation of magnitude between successive elements, it *is* possible to arrange all the positive rational numbers in a sequence r_1, r_2,... similar to that of the natural numbers. Such a sequential arrangement of a set of objects is called an *enumeration* of it, and the set is then called *enumerable*. By exhibiting such an enumeration, Cantor established the equivalence of the set of positive rational numbers with the set of natural numbers, since then the correspondence

$$\begin{array}{ccccc} 1 & 2 & 3 \ \dots & & n \ \dots \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ r_1 & r_2 & r_3 \ \dots & & r_n \ \dots \end{array}$$

is biunique. How is this enumeration obtained?

Every positive rational number can be written in the form *p/q*, with *p* and *q* natural numbers. We place all these numbers in an array, with *p/q* in the p^{th} column and q^{th} row: for example, 5/7 is assigned to the fifth column and the seventh row. All the positive rational numbers may now be arranged in a sequence along a continuous "zigzag" line through the array defined as follows. Starting at 1, we proceed to the next entry on the right, obtaining 2 as the second member of the sequence, then diagonally down the left until the first column is reached at entry 1/2, then vertically down one place to 1/3, diagonally up until the first row is reached again at 3, across to 4, diagonally down to 1/4, and so on, as shown in the figure below. Proceeding along this zigzag line we generate a sequence 1, 2, 1/2, 1/3, 2/2, 3, 4, 3/2, 2/3, 1/4, 1/5, 2/4, 3/3, 4/2, 5, ... containing all the positive rationals in the order in which each is met along the line. In this sequence we now delete all those numbers *p/q* for which *p* and *q* have a common factor, so that each rational number will appear exactly once and in its simplest form. In this way we obtain a sequence 1, 2, 1/2, 1/3, 3, 4, 3/2, 2/3, 1/4, 1/5, 5, ... in which each rational occurs exactly once. This shows that the set of positive rational numbers is enumerable, and it is easy to deduce from this that the set of all rational numbers—positive, negative, or zero—is also enumerable. For if r_1 , r_2, ... is an enumeration of the positive rational numbers, then the sequence 0, r_1, $-r_1$, r_2, $-r_2$, ... is an enumeration of all the rational numbers.

1	2	3	4	5	6	7 …
1/2	2/2	3/2	4/2	5/2	6/2	7/2 …
1/3	2/3	3/3	4/3	5/3	6/3	7/3 …
1/4	2/4	3/4	4/4	5/4	6/4	7/4 …
1/5	2/5	3/5	4/5	5/5	6/5	7/5 …
1/6	2/6	3/6	4/6	5/6	6/6	7/6 …
1/7	2/7	3/7	4/7	5/7	6/7	7/7 …
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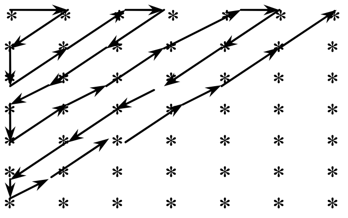

The fact that the rational numbers are enumerable might lead one to suspect that *any* infinite set is enumerable, thus bringing our tour of the infinite to a speedy conclusion. But nothing could be further from the truth. For Cantor also proved the astonishing result that the set of all *real* numbers, rational and irrational, is *not enumerable*. In other words, the totality of real numbers is a radically different, indeed a *larger*, infinity than are the totalities of natural or rational numbers. The idea of Cantor's proof of this fact is to exhibit, for any given enumerated sequence s_1, s_2, ... of real numbers, a new real number which is *outside* it. As an immediate consequence, no given sequence s_1, s_2, ... can enumerate all the real numbers, so that their totality is not enumerable.

Recall that any real number may be regarded as an infinite decimal of the form

$$N.\ a_1a_2a_3.....$$

where *N* denotes the integral part and the small letters the digits after the decimal point. Now suppose we are given an enumerated sequence s_1, s_2, s_3, ... of real numbers, with

$$s_1 = N_1.\ a_1a_2a_3....$$
$$s_2 = N_2.\ b_1b_2b_3....$$
$$s_3 = N_3.\ c_1c_2c_3....$$

We proceed to construct, using what has become known as *Cantor's diagonal process*, a new real number which we show does not occur in the given sequence. To do this we first choose a digit a which differs from a_1 and is neither 0 nor 9 (to avoid possible ambiguities which may arise from equalities like 0.999... = 1.000...), then a digit b different from b_2 and again unequal to 0 or 9, similarly c different from c_3 and so on down the "diagonal" of the above array. (For example, we might choose a to be 1 unless a_1 is 1, in which case we choose a to be 2, and similarly down the line for all the digits $b, c, d, \ldots$.) Now consider the infinite decimal

$$x = 0.abcd....$$

This new real number x then differs from any one of the numbers $s_1, s_2, s_3, \ldots$: it cannot be equal to s_1 because the two differ at the first digit after the decimal point; it cannot be equal to s_2 because these two differ at the second digit after the decimal point, and so on. Accordingly x does not occur in the given sequence, so that no given sequence can enumerate the totality of real numbers, which is therefore nonenumerable.

Recalling that the set of real numbers represents the *geometric continuum*, that is, the totality of points on a line, we infer from Cantor's theorem that this latter set is also nonenumerable.

This theorem allows us to draw some further striking conclusions. Let us call a real number *algebraic* if it is a root of an algebraic equation of the form

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0,$$

where the a_i are integers. For example, the real numbers $\sqrt{2}$, $\sqrt[3]{5}$ are algebraic. A real number which is not algebraic is called *transcendental.* Do transcendental numbers exist? Cantor's theorem provides the startling answer that, not only do transcendental numbers exist, but there are *nonenumerably many* of them. For it can be shown that the set of algebraic numbers is *enumerable*: indeed, each algebraic number can be correlated with the finite sequence of integer coefficients in the algebraic equation of which it is a root, and a simple extension of the argument used to prove the enumerability of the rationals shows that the set of finite sequences of integers is enumerable. From this it follows that the set of transcendental numbers is nonenumerable. For, if it were enumerable, then the set of all real numbers, as the union of the set of transcendental numbers with the enumerable set of algebraic numbers, would itself be enumerable, contradicting Cantor's theorem.

This conclusion is remarkable in that it shows that "most" real numbers are transcendental *without exhibiting a single one*. Actually the existence of transcendental numbers was established before Cantor's theorem by *J. Liouville* (1809–1882), who provided explicit examples of them, for instance, the infinite decimal

$$0.101001000000100000000000000000000000001.....$$

where the number of zeros between ones increases as $n!$. In 1873 *Charles Hermite* (1822–1901) established the transcendentality of the number e (defined on p. 161) and, as already mentioned, in 1882 Lindemann established the transcendentality of π. In the nineteen thirties *A .O. Gelfond* (1906–1968) showed that numbers like $2^{\sqrt{2}}$ are also transcendental[2].

For set theory the importance of Cantor's theorem stems from the fact that it reveals the presence of at least *two* types of infinity: the enumerable infinity of the set of natural numbers and the nonenumerable infinity of the set of real numbers or the geometric continuum. We shall now extend the use of the term "number" to *arbitrary* —even *infinite*—sets by saying that two equivalent sets, whether they be finite or infinite, have the *same cardinal number.* If *A and B* are finite this of course reduces to the statement that they have the same (natural) number of elements. Let us term *infinite* the cardinal number of an infinite set. We have seen that any infinite set is equivalent to each of its subsets obtained by the deletion of one member, so that adding this one element to the subset does not change its cardinal number. This means that, *for any infinite cardinal number* **k,**

$$\mathbf{k} + 1 = \mathbf{k}.$$

It is the fact that infinite cardinal numbers satisfy this equation that distinguishes them from ordinary integers.

We shall further say that a set A has a *greater cardinal number* than a set B if B is equivalent to a subset of A, but A is not equivalent to B or any of its subsets. Since the set of natural numbers is a subset of the set of real numbers, while, as we have seen, the latter is not equivalent to the former nor to any of its subsets, it follows that the set of real numbers has a greater cardinal number than the set of natural numbers. Cantor went further and established the general fact that, *for any set A*, *it is possible to produce another set B with a greater cardinal number*. The set B is chosen to be what we have called in Chapter 6 the *power set* of A, that is, the set of all subsets of A, including A itself and the empty subset $\varnothing$ which contains no elements at all. Thus each element of B is *itself* a set, comprising certain elements of A. Now suppose that B were equivalent to A or to some subset of it, that is, there is a biunique correspondence $a \mapsto S_a$ between the elements of A (or a subset of it) and the elements of B, i.e. with the subsets of A, where we denote by S_a the subset of A correlated with the element a of A. We show that this is impossible by exhibiting a subset T of A, i.e. an element of B, which *cannot* have any element of A correlated with it. We obtain T as follows. For any element x of A which is correlated with an element S_x of B, we have two possibilities: either x is a member of S_x, or it is not. We define T to be the subset of A consisting of all correlated elements x of A for which x is *not* an element of S_x. We can now show that T is not correlated with any element of A, i.e. that T does not coincide with S_x for any element x of A. For suppose that T and S_a *did* coincide for some specific element a of A. Then, since T consists of all elements x of A for which $x \notin S_x$, it follows, in particular, taking x to be a, that $a \in T$ exactly when $a \notin S_a$. But since T has been assumed to be identical with S_a, $a \in T$ precisely when $a \in S_a$. Thus a is not in S_a exactly when a is in S_a, a contradiction. From this contradiction we conclude that our original assumption was incorrect, and it follows that T cannot coincide with any S_x.

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[2] In 1934 Gelfond proved the general result that a^b is transcendental whenever $a \neq 0, 1$ is algebraic and b is irrational and algebraic.

This argument establishes the impossibility of setting up a biunique correspondence between the elements of A, or one of its subsets, and those of its power set B. On the other hand, the pairing $x \mapsto \{x\}$, where, for each x in A, $\{x\}$ denotes the subset of A whose sole element is x, defines a biunique correspondence between A and the subset of B consisting of all one-element subsets of A. It follows from the definition that B has a greater cardinal number than A.

Thus Cantor showed that, just as there is no greatest integer, so there is no greatest infinite cardinal number, which means that the infinite cardinal numbers form an unbounded ascending sequence. Cantor used the symbol "$\aleph$"—*aleph,* the first letter of the Hebrew alphabet—to denote the members of this sequence, which is then written $\aleph_0$, $\aleph_1$, $\aleph_2$, … . Its first member, $\aleph_0$, is the cardinal number of the set of integers. It is a surprising fact that, not only is the cardinal number of the set of points on a line segment independent of its *length*, but the cardinal number of the set of points in a geometric figure is independent of its *number of dimensions*. Thus, for example, we can show that the cardinal number of the set of points in a square is no greater than that of the set of points on one of its sides. This is done by setting up the following correspondence.

If (x, y) is a point of the unit square (i.e., with $0 \leq x, y \leq 1$), x and y may be written in decimal form as

$$x = 0.a_1a_2a_3a_4....$$
$$y = 0.b_1b_2b_3b_4....$$

To the point (x, y) we then assign the point

$$z = 0.a_1b_1a_2b_2a_3b_3a_4b_4....$$

on the bottom side of the square, obtained by interlacing the two decimals corresponding to x and y. Clearly different points (x, y), (x', y') in the square correspond to different points z, z' on the side, so that we have a biunique correspondence between the set of points in the square and a subset of the set of points on the side. Thus the cardinal number of the set of points in the square does not exceed the cardinal number of the set of points on a side. A similar argument shows that the cardinal number of the set of points in a cube is no greater than that of the set of points on an edge. By slightly refining these arguments it can be shown that the cardinal numbers of these various sets actually *coincide.*

Despite the fact that these results seem to contradict the intuitive idea of dimension, it must be pointed out that the correspondence we have set up is not a *topological* equivalence: it does "tear points apart". Travelling smoothly along the segment from 0 to 1, we find that the corresponding points in the square do not form a smooth curve, but instead form an entirely random pattern. The dimension of a set of points in fact depends primarily on the way its elements are distributed in space, rather than on its cardinal number. This follows from Brouwer's theorem—mentioned in Chapter 8—that the dimension of a region in Euclidean space is a topological invariant.

We have seen that the cardinal number **c** of the set of real numbers exceeds that of the set of integers. It is natural to ask: by *how much* does the one exceed the other? That is, *how many* cardinal numbers are there (strictly) between that of the set of integers and that of the continuum? The simplest possible answer is: *none.* The conjecture that this is the correct answer is called the *continuum hypothesis.* Since $\aleph_1$ is by definition the next largest cardinal number above $\aleph_0$, the cardinal number of the set of integers, this hypothesis may be succinctly stated in the form

$$\mathbf{c} = \aleph_1.$$

Cantor firmly believed in its truth and expended much effort in attempting, without success, to prove it. It was not until 1963 that *Paul J. Cohen* showed that this hypothesis is in fact *independent* of the principles on which set theory is built (*Kurt Gödel,* 1906–76, having shown in 1938 that it is consistent with these principles). In fact Cohen showed that **c** could coincide with virtually *any* cardinal number not less than $\aleph_1$: the principles of set theory simply do not fix an exact value for **c** in terms of its position in the scale of alephs. Thus the status of the continuum hypothesis in set theory bears a certain resemblance to that of Euclid's fifth postulate in geometry: just as the fifth postulate can be denied so as to produce a noneuclidean geometry, so the continuum hypothesis can be denied so as to produce a "noncantorian" set theory.

Infinite sets, although somewhat paradoxical in character, are, as we have come to realize, not contradictory in themselves. As originally formulated, however, set theory *does* contain contradictions, which result not from admitting infinite totalities *per se,* but rather from countenancing totalities consisting of *all* entities of a certain abstract kind. This is best illustrated by the infamous *Russell paradox*, discovered in 1901 by the philosopher-mathematician *Bertrand Russell* (1872–1970).

Russell's paradox arises in the following way. It starts with the truism that any set is either a member of itself or not. For instance, the set of all cats is not a member of itself since it is not a cat, while the set of all non-cats is a member of itself since it is a non-cat. Now consider the set consisting precisely of all those sets which are *not* members of themselves: call this set *R*. Is *R* a member of itself or not? Suppose it is. Then it must satisfy the defining condition for inclusion in *R,* i.e. it must *not* be a member of itself. Conversely, suppose it is not a member of itself. Then it *fails* to satisfy the defining condition for inclusion in *R,* that is, it *must be* a member of itself. We have thus arrived at the unsettling, indeed contradictory, conclusion that *R* is a member of itself precisely when it is not. We note that whether *R* is finite or infinite is irrelevant; the argument depends solely on the defining property for membership in *R.*

Russell's paradox also appears when we consider such curious entities as, for instance, the bibliography of all bibliographies that fail to list themselves: such a bibliography would, if it existed, list itself precisely when it does not. In this case, however, we may simply infer that the entity in question does not exist, a conclusion we cannot draw in the case of the Russell set *R* without bringing into question the very basis on which sets have been introduced.

Russell's paradox has a purely *linguistic* counterpart known as the *Grelling-Nelson paradox*. Call an (English) adjective *autological* if it is true of itself and *heterological* if not. For instance, the adjectives "polysyllabic", "English" are autological, and "palindromic", "French" are heterological. Now consider the adjective "heterological". Is it autological or not? A moment's thought reveals that it *is* precisely when it *is not*..

Another principle of set theory whose enunciation (in 1904 by *Ernst Zermelo,* 1871–1953) occasioned much dispute is the so-called *axiom of choice.* In its simplest form, the axiom asserts that, if we are given any collection *S* of sets, each of which has at least one member, then there is a set *M* containing exactly one element from each set in *S*.[3] No difficulty is encountered in assembling *M* when there are only finitely many sets in *S*, or if *S* is infinite, but we possess a *definite rule* for choosing a member from each set in *S*. The problem arises when *S* contains infinitely many sets, but we have *no rule* for selecting a member from each: in this situation, how can the procedure be justified of making infinitely, perhaps even nonenumerably, many arbitrary choices, and forming a set from the result?

The difficulty here is well illustrated by a Russellian anecdote. A millionaire possesses an infinite number of pairs of shoes, and an infinite number of pairs of socks. One day, in a fit of eccentricity, he summons his valet and asks him to select one shoe from each pair. When the valet, accustomed to receiving precise instructions, asks for details as to how to perform the selection, the millionaire suggests that the left shoe be chosen from each pair. Next day the millionaire proposes to the valet that he select one *sock* from each pair. When asked as to how *this* operation is to be carried out, the millionaire is at a loss for a reply, since, unlike shoes, there is no intrinsic way of distinguishing one sock of a pair from the other. In other words, the selection of the socks must be truly arbitrary.

One curious consequence of the axiom of choice is the *paradoxical decomposition of the sphere*, formulated in 1924 by the Polish mathematicians *Stefan Banach* (1892–1945) and *Alfred Tarski* (1902–1983). In one form, it asserts that a solid sphere can be cut up into finitely many (later shown to be reducible to five!) pieces which can themselves be reassembled exactly to form *two* solid spheres, *each* of the same size as the original. Another version of this "paradox" is that, given any two solid spheres, either one of them can be cut up into finitely many pieces which can be reassembled to form a solid sphere of the same size as the other. Thus, for example, a sphere the size of the sun can be cut up and reassembled so as to form a sphere the size of a pea! Of course, the phrase "can be cut up" here is to be taken in a metaphorical, not practical, sense; but this does not detract from the counterintuitiveness of these results. Strange as they are, however, unlike Russell's paradox, they do not constitute outright contradictions: sphere decompositions become possible in set theory only because continuous geometric objects have been decomposed into discrete sets of points which can then be rearranged in an arbitrary manner.

The perplexities surrounding the emergence of set theory are collectively designated by historians of mathematics as the *third crisis* in the foundations of mathematics (the two previous being the Pythagorean discovery of incommensurables, and the shaky state of the foundations of the calculus in the seventeenth and eighteenth centuries). Attempts to resolve this crisis took several different forms, all of which involved subtle analysis of the nature of mathematical concepts and reasoning—topics

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[3] Implicit use of the axiom of choice was made in our proof that every infinite set is equivalent to a proper subset of itself. It can be shown that this use is essential.

in *mathematical logic* and the *philosophy of mathematics* which we take up in the final chapter.

The purely technical difficulties in set theory were overcome when, in the first few decades of this century, it was *axiomatized* (by Zermelo and others) in such a way as to circumvent contradictions such as Russell's paradox by suitably restricting the formation rules for sets. Any residual doubts concerning the acceptability of the axiom of choice were dispelled in 1938 when Gödel established its consistency with respect to the remaining axioms of set theory. These developments enabled the majority of mathematicians to accept set theory as providing an adequate foundation for their work. Mathematicians find set theory acceptable not solely for the practical reason that it enables mathematics to be *done,* but also because it is consonant with the unspoken belief of the majority that mathematical objects actually *exist* in some sense and mathematical theorems express truths about these objects. This is a version of the philosophical doctrine of *Realism*, also termed, with less accuracy, *Platonism.*

Hermann Weyl has written:

> ‘Mathematizing’ may well be a human creative activity, like music, the products of which not only in form but also in substance are conditioned by the decisions of history and therefore defy complete objective rationalization.

This is a perceptive observation; nevertheless, the attempt to explicate the mathematical infinite, and so to grasp the ultimate nature of mathematics, has borne much fruit and will no doubt long continue to be a source of inspiration.